Chapter 4

Guo Shoujing and the Shoushi Li—Reconstructing the Solar Calendar

Guo Shoujing (1231–1316) was born in Xingtai, in the Hebei region of northern China, during the waning years of the Jin dynasty, just before the final collapse of the Southern Song. He grew up in a period of profound political upheaval and cultural transformation, that accompanied the Mongol Empire expansion southward.

From an early age, Guo displayed remarkable aptitude in mathematics, astronomy, and hydraulics. He was tutored by the mathematician and astronomer Li Zhi (not to be confused with the later philosopher), under whose guidance he mastered classical Chinese mathematical texts, including the Nine Chapters on the Mathematical Art. He also studied the observational techniques and calendrical systems used in Tang and Song dynasty astronomy. His education combined rigorous empirical practice with algorithmic thinking, reflecting the Chinese scholarly tradition of computation and measurement over metaphysical speculation.

The collapse of the Song dynasty created a new intellectual climate in which technical skill, especially in service of statecraft, was in high demand. Guo's early work on water conservancy projects and canal engineering brought him to the attention of Yuan officials[1]. His ability to design efficient irrigation and transportation systems earned him commissions from the Yuan court, which was eager to consolidate control over a fragmented and diverse empire. Despite his Han Chinese heritage and the ambivalence many intellectuals felt toward serving foreign rulers, Guo chose a path of pragmatic cooperation, applying his expertise to rebuild infrastructure and establish scientific institutions.

This pragmatic service would culminate in his leadership of the Yuan dynasty's astronomical reform, including the construction of observatories across the empire and the development of the Shoushi Li calendar—one of the most accurate solar calendars of the pre-modern world.

Three characters represent the words Shoushi Li. The meaning of the character Shou is teaching. The meaning of the character shi is timing. The meaning of the character Li is calendar. Putting the three characters together yields "Teaching Timing Calendar". To those unfamiliar with Chinese these three words put together seem to convey something, but whatever that might be, it seems uncertain.

A Chinese person explains; Chinese characters can be thought of as acronyms. The letters LOL don't mean anything to one unfamiliar with the use of English language acronyms, even if they know the letters. That's because there is more to LOL than the letters.

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[1] The Yuan dynasty refers to the dynastic empire established by Genghis Khan. Yuan officials are officials in the Yuan government. Due to shortcomings in Mongolian education, Genghis Khan was illiterate, officials were often not Mongolian.

Similarly, the characters “Teaching Timing Calendar” have a meaning beyond the words. For centuries, the Chinese agrarian population desired formal announcements of important dates for planting and harvesting seasons. Additionally, formalization of holidays and tax collection dates were also critical. The characters “Teaching Timing Calendar” describe the calendar’s functionality toward official presentation of the dates. Literally the calendar taught the population when to plant, when to harvest, and when the holiday and taxation days occurred among other important dates. For those who, prior to the explanation, found the name without meaning and somewhat strange (this includes the biological author), LOL.

After providing a brief introduction to Chinese astronomy, this chapter presents Guo’s development of the Shoushi Li calendar as a data science problem. The problem connects well with Aristarchus’ parametric model. Aristarchus uses a minimal set of observations to directly measure the parameters of his parameteric model. Guo Shoujing takes many measurements into account and from these many maeasurements, must choose the parameters that best fit the data.

4.1 Contrasts: Western and Chinese Approaches to the Heavens

This section contrasts the perspective of Greek and Chinese astronomers toward uncovering the pathways of heavenly bodies. The section begins with the ancient Greeks. As the previous chapter presents the social and political environment of the Greek astronomers, this chapter only presents a history of the technical pathway. Moving on to Chinese astronomy, in addition to the technical approach, the section describes political and social background information.

The ancient Greeks, particularly Ptolemy (c. 100–170 CE), focused on mathematical elegance and theoretical sufficiency. The two united into a idealization. The roots of the approach can be traced to Plato (c. 428–348 BCE), who established the philosophical ideal that celestial motion must be described in terms of uniform circular motion. In his cosmological vision, the heavens embodied divine perfection, and the circle was its purest representation. However, observed planetary motions often deviated from simple circular paths. This obsession with the ideal circle prompted the geometric approach which also spawned new mathematical methods.

To reconcile observation with Plato’s ideal, Eudoxus of Cnidus (c. 408–355 BCE) proposed a system of nested spheres—each celestial body was carried on a series of rotating concentric spheres, whose combined motion could approximate the irregularities seen in the heavens. This approach preserved circularity but was limited in predictive accuracy.

A major advance occurred at the Alexandrian school, where astronomers had access to centuries of Babylonian and Greek observations. Hipparchus (c. 190–120 BCE), working in the 2nd century BCE, introduced the concept of the epicycle: a small circle on which a planet moves, which itself revolves along a larger circle (the deferent) centered near the Earth. This geometric innovation dramatically improved the accuracy of planetary models.

Later refinements introduced by Ptolemy included the equant point—a location offset from the center of the deferent, from which the angular motion of the epicycle’s center appeared uniform. This violated uniform circular motion relative to Earth but allowed for more accurate modeling. Ptolemy also repositioned the deferent’s center away from Earth to further improve the model.

These innovations required the determination of parameters which in turn lead to new developments for solving complicated equations. As with Aristarchus, Ptolemy generally used a minimal set of observations to determine model parameters, even though earlier records provided many observations. Once entering the parameter set into the equations, Ptolemy approximated solutions using novel iterative processes that are now a standard part of numerical procedures.[2]

An exception was in his calculation of the length of the planetary years, where he compared historical observations recorded at Alexandria with his own measurements. In this regard the Greek astronomer and the Chinese astronomers used similar methods. Otherwise, his methodology remained minimalist and primarily deterministic.

Ptolemy's description of the motions of all heavenly bodies prevailed for centuries until Copernicus upended the geocentric vision. Nevertheless, Copernicus used Ptolemy's methods to discern the pathways of the planets about the sun from a heliocentric perspective. Deferents, epicycles, and equants remained a feature of the models of planetary motion. Using only the required number of observations necessary to fit model parameters, proved inadequate. The approach broke down as predictions diverged from actual events.

In contrast, Chinese astronomy emphasized empirical adequacy and practical computation. Rather than minimizing data, Chinese astronomers collected extensive observational records over centuries and focused on algorithms that reproduced the patterns embedded in the data. Predictive accuracy took precedence over theoretical unification.

Guo Shoujing (1231–1316) exemplifies this tradition. His method was not to select a small number of data points to fit parameters, but rather to use a vast number of historical observations to refine interpolative formulas that could track solar motion with high accuracy over time.

Astronomy in imperial China was deeply institutionalized but rarely glorified in cultural terms. Mathematicians and astronomers were bureaucrats, not literati; their social rank was often lower than that of Confucian scholars and poets. While poetry was celebrated as a noble art, mathematical skill was considered utilitarian.

All bureaucrats were required to pass the civil service examination to obtain official positions. This exam tested knowledge of Confucian texts and classical Chinese literature, and it included a required poetry submission. Although the literary component dominated, the examination also included a mathematical section, largely for practical reasons related to taxation and administration.

Despite the lack of cultural prestige, astronomy received strong state support because it was essential for calendar-making—a critical tool for taxation, agriculture, and imperial legitimacy. The calendar was a symbol of cosmic order, and any inaccuracies could be politically dangerous.

Along with the Mongol conquest of China came a governmental change in attitude toward astronomers and mathematicians. Genghis Khan (c. 1162–1227), was illiterate and had no appreciation for the art of Chinese poetry. His grandson, Kublai Khan, shared the father's lack of appreciation for literary arts, but did have a fascination with mathematics and its application to solving problems.

Brutality and the resulting widespread resentment marked the Mongolian conquest. The Mongols razed cities, decimated populations, and dismantled institutions. When Genghis Khan established the Yuan dynasty, many Han Chinese scholars faced a difficult choice: resist or cooperate. Guo Shoujing, known for his technical brilliance, was given a commission under Kublai Khan (1215 – 1254). initially as an engineer. Like many of his compatriots, Guo likely harbored complex feelings toward serving the conquerors, but he made do with the circumstances.

As an engineer, Guo oversaw the construction of canals, which were essential for transportation and irrigation. Later, he was tasked with mapping the southern boundaries of the expanding Yuan empire. Ultimately, he was appointed to a high-ranking astronomical post and oversaw the construction and operation of 27 observatories across the empire. These observatories were central to the development of a new calendar that the Yuan dynasty

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[2] Kepler confronts complicated equations and follows Ptolemy down the iterative path (see chapter 5).

implemented across their diverse territories. Guo was instrumental in designing both the observatories and the calendar system.

Guo Shoujing's innovations built upon a long and rich tradition of empirical astronomy and algorithmic computation in China.

- **Gan De** (4th century BCE) and **Shi Shen** were early pioneers in systematic star cataloging and planetary observations. They recorded positions and motions of celestial bodies and introduced rudimentary predictive techniques.
- **Luoxia Hong** (2nd century BCE) was the first Chinese astronomer known to propose a reasonably accurate solar year value of 365.25 days.
- **Liu Hui** (c.225–295) made foundational contributions to numerical approximation and calendrical mathematics.
- **Zu Chongzhi** (429–500) refined the length of the tropical year to approximately 365.24281481 days—remarkably close to the modern value.
- **Shen Kuo** (1031–1095) made significant strides in observational astronomy and theory.
- **Su Song** (1020–1101) constructed an elaborate astronomical clock tower in Kaifeng.

Guo's methods are not readily available. Perhaps they have been lost during the chaos of Yuan's dynastic collapse earlier algorithms for interpolation (notably in the *Dayan li* system) laid the groundwork. These included piecewise interpolation schemes using linear or quadratic segments to approximate irregular solar motion. Guo extended these methods by formalizing them into multi-term recurrence relations that spanned entire calendar segments.

4.2 The Observer

The previous chapter first presents Aristarchus' model, and then discusses observational methodologies. This sequence reflects Aristarchus' mindset; follow the seductive theory, as for the observations, a detail left to others. In this chapter, reflecting the centrality of observations to Guo Shoujing's development of the calendar, we begin with observations.

From a young age, Guo Shoujing exhibited remarkable mechanical aptitude. As a child, he constructed a functioning water clock—a device that regulated the flow of water between containers to mark the passage of time. The precision and ingenuity required for such an instrument were unusual for someone so young and revealed an early mastery of both practical craftsmanship and abstract measurement. This early water clock was not merely a toy or curiosity; it signaled a deeper engagement with the principles of timekeeping and engineering that would later manifest in his large-scale projects for the Yuan dynasty, including the construction of hydraulic systems, astronomical instruments, and calendar mechanisms.

When he was appointed to oversee the construction of the Yuan dynasty's astronomical infrastructure, Guo directed the building of 27 observatories throughout the empire. He equipped each with standardized instruments, some of which were his own design. Chief among these was the gnomon.

The gnomon, a vertical rod or pillar used to cast shadows, was one of the oldest tools in Chinese astronomy, yet Guo Shoujing brought renewed rigor and precision to its design and application. Central to the creation of a reliable solar calendar was the need to track the sun's apparent motion with precision. Repeated measurements around the solstices allowed Guo to determine the sun's highest and lowest noonday altitudes with great accuracy—essential data for refining the length of the tropical year and anchoring the solar terms that structured the traditional Chinese calendar. Measurements were made not just solstices and equinoxes, but every day throughout the year. By recording the length of the gnomon's shadow at local noon each day, Guo and his assistants could determine the sun's meridian altitude. These daily measurements allowed for the continuous refinement of solar longitude calculations and the accurate prediction of seasonal markers.

Even small errors in shadow measurement could introduce cumulative discrepancies in calendrical reckoning, leading to misaligned agricultural activities or state rituals. To enhance accuracy, Guo Shoujing constructed his gnomons with exceptional care. He increased their height to improve angular resolution; taller gnomons cast longer shadows, allowing for finer measurement divisions on the ground plane. He also ensured that the base was meticulously leveled and aligned along a true north-south meridian, minimizing systematic errors due to tilt or misalignment. The shadow was projected onto a carefully prepared horizontal surface marked with a scale that enabled readings with a precision unprecedented in earlier Chinese instruments.

At the Gaocheng observatory in Henan Province, a key site in Guo's network, the gnomon stood approximately 12.6 meters tall. It cast shadows onto a meticulously leveled stone ground plane—a meridian line—extending 31.2 meters due north from the base. This long baseline enabled high-resolution measurements of the shadow's position throughout the year.

But this gnomon had an add-on feature unavailable to any other astronomer in the world. Although cheap to produce and unimpressive looking, this device solved a problem common to many tall gnomons. As the Sun drops lower around the Winter solstice, and the gnomons shadow lengthens, the shadow tip becomes difficult to discern. To overcome this Gou Shoujing proposed no more than a plate that rotates on a stand with a pinhole at its center; essentially a pinhole camera.

A technician would place the stand on the gnomon shadow where the shadow begins to fade. The technician would then rotate the plate, aligning the the incoming solar rays with the pinhole, until a nice distinct image of the gnomon's tip would appear on the ground. The pinhole camera brought the shadow's tip into focus.

Aside from the gnomon, Guo Shoujing's observatories were equipped with a variety of specialized instruments designed to expand the range and precision of astronomical observations. Among the most significant was the *Simplified Armillary Sphere*. This instrument was a streamlined and more stable variant of the traditional armillary sphere, retaining only the essential meridian and equatorial circles. By reducing mechanical complexity, Guo enhanced the instrument's accuracy and durability, making it especially effective for measuring celestial altitudes and right ascensions.

Another crucial instrument was the *Clepsydra* (*louhu*), or water clock. These timekeeping devices were carefully calibrated to ensure consistent flow and were indispensable for recording the timing of celestial events throughout the night. With their help, observers could track the motion of stars and planets with reliable temporal resolution, a key requirement for compiling long-term astronomical datasets.

The *Horizon Circle* also featured prominently in Guo's observational arsenal. This instrument was specifically designed for angular measurements near the horizon, making it ideal for determining the precise moments of sunrise and sunset—critical data points for anchoring the solar calendar.

These instruments enabled a wide array of observations essential to the construction of the solar calendar. Guo's teams tracked the sun's daily position relative to the horizon, logged the precise moments of equinoxes and solstices, and recorded stellar transits across the meridian. With multiple observatories operating across different latitudes, Guo was able to cross-check and interpolate results, eliminating local anomalies due to atmospheric refraction or terrain.

To ensure that observations from across the empire could be meaningfully compared, Guo standardized procedures, calibration methods, and reporting formats at all 27 observatories. Each site followed the same protocols, used similarly constructed instruments, and recorded data in a consistent manner. This system-wide uniformity allowed for precise cross-checking and merging of observations from different regions. Over time, the accumulation of this carefully curated data became the source for the development of the most precise calendar in the world.

4.3 From Observations to Forecast

Observations were central, but only a means to an end. The end is a clearly defined process for creating a lunar calendar and forecasting solar driven events, for example winter/summer solstices and spring/fall equinoxes and solar/lunar eclipses, on lunar driven dates. This requires models of both solar and lunar motion and the melding of the two. Additionally, the Shoushi Li project addresses the motion of planets and positioning of stars. We'll call it extensive, but that is an understatement.

Needless to say, the task of a comprehensive review of methodology is far too Herculean for the writers to write as well as most readers to read. For the purpose of examining the project as a data science project, we narrow our effort to two facets of the Shoushi Li,

- determination of the length of a solar year, and
- the apparent annual motion of the Sun.

With respect to these two facets, this chapter describes the process that Guo Shoujing employed to transform a rich observation set into predictions.

As with the presentation of Aristarchus' work, we focus on the main thrust and forego complexity. Unlike Aristachus' treatise, documentation of Guo Shoujing's methods is lacking. For this reason, our goal requires a bit of conjecture. But the conjecture is not fantasy. We are piecing together a viable pathway toward the calendar's development using the documented evidence that is available. It is a plausible reconstruction.

The guiding philosophy of our reconstruction is to use the most straight forward procedures that had been used by previous astronomers. This philosophy yields a different reconstruction than one frequently encounters in a literature search; more on that later. We first provide the information sources from which we create our reconstruction. Then we get on with it.

Only Three Sources

Following the tradition of previous dynasties, the Yuan dynasty under Kublai Khan dedicated itself to documenting its undertakings. Documentation certainly underwent revisions to assure the calibration of politically sensitive material with court approved political philosophy. In the case of the technical material behind the development of the Shoushi Li, this should have been but a minor issue – particularly as Kublai Khan himself formally authorized the project in 1276.

The technical work fell to Wang Xun and Guo Shoujing who supervised a team of astronomers and technicians. They reported to bureaucrats with impressive titles. The team of astronomers issued two tranches of reports. Under pressure to show progress, Wang Xun issued the first set of reports in December of 1280. The technical information in these reports was sufficient for the issuing of a calendar; 1281 marks the year of the first Shoushi Li calendar.

This first issuance garnered the continued support of the project. Wang Xun passed away in 1284. While the official Assistant Supervisor, Guo Shoujing, did not receive the title of Director until 1287, he was for all intents and purposes the acting director in the interim. Upon official promotion, Guo Shoujing convinced Kublai Khan to support the construction of additional observatories which resulted in Guo Shoujing overseeing a network of observatories across what is now modern day China and Korea.

Guo Shoujing compiled a set of documents with pages that numbered in the thousands. As with Wang Xun's initial reports, they have vanished. Perhaps they lay alongside the graves of Genghis and Kublai Khan, which have also disappeared. We leave it to the reader to determine which of these losses is more consequential.

Without direct documentation from the originators, what are we left with? Needham and Nathan Sivin are the West's best guides. Both are authorities in Chinese culture with a keen interest in Chinese science. Both are capable translators of traditional Chinese scientific documents. Both have had access to Chinese archives and both site the same sources.

There are two predominant sources dating from the early 14th century, within 50 years of Guo Shouying's second tranche of documents. The Ming dynasty which overtook the Mongolian Yuan dynasty sponsored a review of the Shoushi Li project and issued the two documents that both Needham and Sivin cite. It is notable that the officials of the Ming Dynasty had access to both Wang Xun's and Guo Shoujing's full set of documents. It is through these officials that we know of their writings.

The first document with translations by Needham and Sivin, *Evaluation of the Shoushi Li*, is a comparative review of the Shoushi Li with its predecessors. From this we recover the method used to determine the winter solstice and determine the length of a year. The second document is *Canon of the Shoushi Li.* This document provides instruction sets for all procedures necessary to predict the Lunar based dates of events of consequence as well as those that may not be so consequential. In particular, the *Canon* contains instructions for the forecasting of the Sun's position at any time of the solar year.

The instruction sets in the *Canon* are for the most part black-box type instructions. They instruct the user to input values into a predetermined set of equations that the *Canon* provides, and execute the underlying arithmetic that yields an output. The *Canon* does not at all address the methodology used to determine the equations along with predetermined constants. That is left to conjecture and we take on this task after describing the methodology used to determine the length of a year.

At first glance, the most promising information source would be *The Account of Conduct of Guo Shoujing*. The account is a memorial of Guo Shoujing's works written by a young astronomer who assisted Guo Shoujing, Qi Luqian and it survives. In his own right, Qi Luqian was a distinguished astronomer who was a worthy collaborator.

Given the technical expertise of Qi Luqian and his first hand experience with the project, one would hope to find some juicy (at least for the mathophile) details of the methodology for positioning the Sun. The energy of that expectation is further caffeinated as the reader notes a section entitled "New Methods".

The portion describing the apparent motion of the Sun as translated by Sivin is three sentences long, with only one containing a description of methodology. We'll get to this sentence later on in the chapter. The other two sentences point to the accuracy of the collected data used in the process and the outcome. There is little juice to squeeze from Luqian's account. Our take away is that no new secret sauce was added to the astronomer's cookbook. This take away reinforces our approach toward the reconstruction; use the most straight forward methods available to Guo Shoujing.

Days in a Year

Sivin's translation of the *Evaluation* includes the following table.

Beyond the face value of the entries, there is additional content in each column.

Astronomer	Year	Time of Winter Solstice	Days from Above Entry	Years from Above Entry	Tropical Year Length
Tsu Ch'ung-chih	462	27.55	-	-	-
Liu Cho	603	46.54	51498.99	141	365.2410
I-hsing	727	36.72	45290.18	124	365.2433
Chou Tsung	1049	45.02	117608.30	322	365.2431
Yao Shun-fu	1099	7.16	18262.14	50	365.2428
Guo Shoujing	1278	45.58	65378.42	179	365.2425

Table 4.1: Historical Measurements of the Tropical Year

The first column gives the name of the contributing astronomer. The name is representative of a team.

The second column gives the year of the observation. The observations span 816 years over several dynasties. Along with the first column this demonstrates a cultural commitment to astronomy and calendar standardization that transcends partisan politics. The table could easily be enlarged to include observations that date back to 250 CE and continue to the present.

The third column gives the day and mark of the occurrence of the solstice from a common reference day of the Chinese lunar calendar. It confuses us as we cannot associate it with a time as one would see in a western system. Nevertheless, along with column 2, column 3 provides the basis for the interpretable values of the fourth column.

The fourth column is unambiguous; the title says it all. The modern day reader may be flummoxed by the entry. It indicates a degree of precision that seems to be beyond what could be performed prior to modern day science. There must be something askew. Who presents days to a second decimal? Nothing is askew. The Chinese used the same modern day decimal system that we use today and those are true entries.

The fifth column is equally unambiguous, subtract the entry from the one above it results in the number of years lapsed. Completion of the fourth and fifth columns produces the climactic moment. Dividing the total number of days lapsed by the total number of years lapsed reveals the number of days in a year, column 6.

For comparison, Liu Xin's first-century estimate was 365.2502 days, and Ptolemy's second-century estimate was 365.24667 days. All the estimates above including those in the table use the exact same procedure. Compared with the modern estimate of 365.24219 days, these are all good estimates.

What makes the estimates so good? The most critical factor is the number of years between the first and final observation. Imagine if the time resolution of days between consecutive solstices was only precise enough to yield whole days. Some years we would record 365 days and other years we would record 366 days. The fraction of years with 366 days reveals the decimal part of the true year length. For example, if the daily spin of the Earth coordinated with its annual revolution such that there is a surplus of exactly $1/4$ spins, then the long term average of the surplus spins would just be .25.

The surplus spin of the Earth is probably not rationally correlated with the Earth's revolution about the sun. The surplus spin is most likely an irrational number. But the logic of the process by which we could come close to that irrational number is the same as that if it were a rational number. As the number of years increases, the ratio of years that our day count is 365 to those in which it is 366 comes closer to the true irrational value, but we need a lot of years to gain accuracy.

The second factor allowing for precision with fewer years is of course, the accuracy of the data. If column 4 in the table is off by a day, then if column 5 is 1000 years, the third decimal point will be incorrect. Yet, Guo

Shoujing's estimate is accurate to the third decimal place, despite being based on just 179 years of data.This is a tribute to the accuracy of both Yao Shun-fu's, and Guo Shoujing's teams.

Finding the Solstice

Around the winter solstice, noontime shadow lengths of a gnomon change very little. Measurement inaccuracies can make it difficult to identify the precise day with the longest shadow. The *Evaluation* illustrates a clever procedure that along with an analysis of other measurements allows Guo Shoujing to determine the day of the winter solstice.

The method exploits symmetry. The Chinese astronomers believed and their data confirms that the Sun's apparent motion is symmetric through the winter solstice. Assume the winter solstice occurs at time zero. If equal time intervals are added and subtracted, the Sun's altitude will be identical. This is a symmetry about the winter solstice. The graph below illustrates the concept.

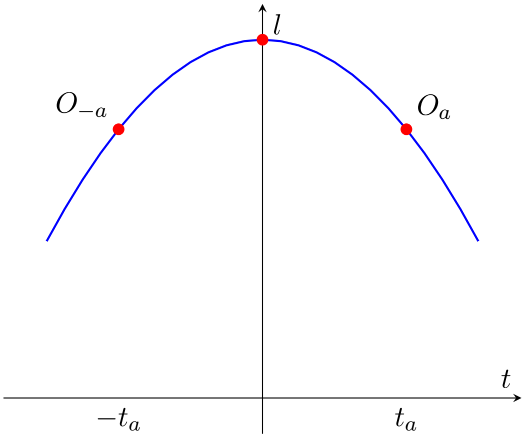

Figure 4.1: Symmetry of shadow length about winter solstice

In the graph, the horizontal axis represents time and the vertical axis represents the length of the gnomon shadow. We set the time of the winter solstice to zero. The red dot at the peak of the curve represents the shadow length at the solstice. Two observations, O_{-a} and O_a are plotted and their respective times are $-t_a$ and t_a. Because the times are symmetric about the time of the solstice, their corresponding shadow lengths are equal.

By analyzing symmetrical patterns in shadow length data as well as additional measurements with other instruments, Guo Shoujing's team was able to pinpoint the winter solstice.

The symmetry about the winter solstice applies to the summer solstice as well; a fact that is quite useful for positioning the sun.

Positioning the Sun

Identifying Earth's position in its orbit around the Sun is central to a heliocentric solar calendar. The angle made by the Earth, Sun and an arbitrary point on the orbit specifies the Earth's position. A geocentric perspective of the solar calendar is similar, just make a switch. Let the Earth be at center and let the sun orbit the earth. Now the angle between the Sun, Earth, and an arbitrary point on the Sun's orbit identifies the position of the Sun on its orbit.

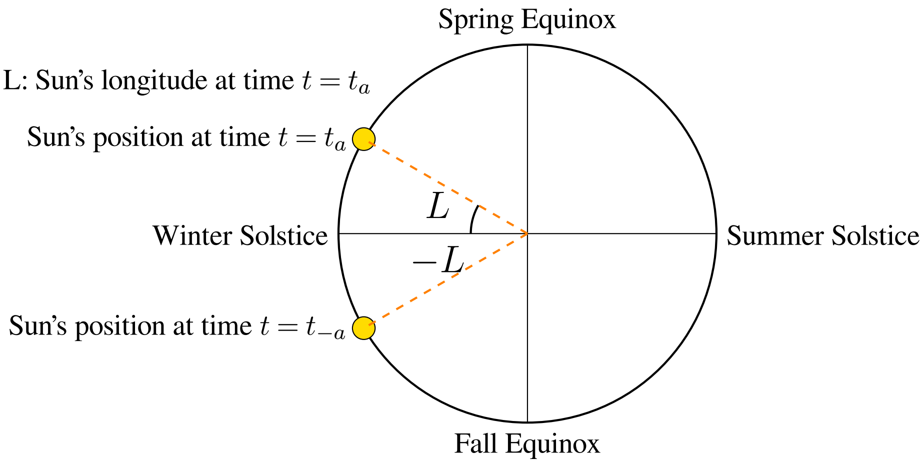

Figure 4.2: Solar Orbit

The Chinese adopted the geocentric view and their arbitrary point was the point on the orbit where the winter solstice occurs. The angle where the winter solstice occurs is zero degrees. When the Sun is at the point where the spring equinox occurs, the angle is $90°$. Further along the Sun's orbit at the point of the summer solstice the angle is $180°$. At $270°$ the sun moves on to the fall equinox before returning to its original position, the winter solstice. The angle is the sun's longitude. As Figure 4.2 indicates, the sun's longitude is the same at each point everyday in both the heliocentric and geocentric settings. Because we are trying to replicate Guo Shoujing's processes, this section adopts his standard which is the geocentric perspective.

Due to the importance of the solstices and equinoxes, Guo Shoujing's team took care to take accurate measurements of their occurrences. The complete methods used to determine these values are beyond the scope of this work. We press on knowing that this information is available.

The problem at hand is to identify the Sun's longitude at any time during the solar year. At our disposal is the concept of symmetry that we previously introduced. Let's explore symmetry a bit more.

Previously we saw that the height of the gnomon's shadow is symmetric in time. Suppose over the course of a year, we take an image of the shadow height each noon. Afterwards we display the figures sequentially either from beginning to end or end to beginning. It will be impossible to determine in which direction the display unfolds. As Figure 4.2 below shows, this time directional symmetry manifests as a spatial symmetry across the Sun's orbit (heliocentric perspective).

Note that the there is a mirror symmetry that cuts across the axis between the winter and summer solstice positions. Guo Shoujing's examination of his data would confirm that this symmetry exists. If we can determine the longitude at any time on the orbit passing from the winter solstice to the summer solstice through the spring equinox, then by symmetry we can determine the longitude at any time on the orbit passing from the summer solstice to the winter solstice through the fall equinox. In other words if we can figure out the upper half of the chart, symmetry provides the lower half. From this point on we consider only the upper half of the Sun's orbit.

Let's take advantage of our symmetry to get one quick and easy result; the time it takes for the Sun to reach the summer solstice from the winter solstice. Symmetry says, it's got to be half the solar year. Guo Shoujing sets the solar year at 365.2425 days. Half the solar year is 182.62125 days. Once again Guo Shagjing's data, which includes measurements of the time of arrival at the summer equinox, would confirm this.

A natural question arises. Is there an equivalent symmetry across the equinoxes? That would further reduce the problem. This issue did occur to Chinese astronomers and their correct conclusion is, no such luck. They base their conclusion on measurements giving the time of occurrence of the equinoxes. If the symmetry does exist, it would take equal time for the Sun to transit from the winter solstice to the spring equinox as it does to transit from the spring equinox to the summer solstice. Guo Shoujing's careful observations show that the times are not even close. The *Canon's* values are 88.91 days to reach the spring equinox and then another 93.71 days to reach the summer solstice[3]

The qualitative picture that emerges is the Sun's longitude moves faster along the path from the winter solstice to the spring equinox than it does along the path from the spring equinox to the summer solstice. Guo Shoujing's goal was to quantify this picture.

To reconstruct Guo Shoujing's method, let's begin with his answer as stated in the *Canon*. There it instructs the reader to determine the longitude between the winter solstice and spring equinox using the following cubic equation.

$$y = 0.051332t - 0.000246t^2 - 0.00000031t^3$$

This proposes a functional relation between time and the longitude. The relation does not come from physical properties. However, a cubic polynomial is quite flexible. It can take many different shapes depending upon the chosen constants, a, b, and c. This is a data fitting exercise.

The *Canon* provides another cubic equation from which one can determine the time required to transit from the spring equinox to the summer solstice. Most likely, the same methodology was used to determine both cubic polynomials and this is our working assumption. For brevity, we reconstruct the methodology that yields the above cubic polynomial.

Before proceeding, it is noteworthy that Guo Shoujing fuses the winter and spring cubic equations together.[4] This separation and fusing process provides additional flexibility for curve fitting purpose. For a given curve, each cubic fits a piece of a curve and fusing the cubics together fits the entire curve quite precisely.The modern phrase for the process is generating a *cubic spline* from the data. Numerical analysts throughout the world employ the method that Chinese mathematicians pioneered. Concerning the above equation, the first task is to understand the variables, y and t. The easy one is t. It is just the time in the number of days in which $t = 0$ represents the time at which the Sun is at the winter solstice. The value y represents a quantity that requires some explanation.

The Chinese astronomers knew that the motion of the Sun around its orbit is close to uniform. That is to say, everyday the Sun increments its longitude by nearly the same number of degrees. Early Chinese astronomers estimate the year's length at 365.25 days. According to the early astronomers, if the motion were perfectly uniform, the Sun would move 360 degrees in 365.25 days, yielding a daily change of about 0.98563 degrees.The quantity y represents the difference between the longitude of a theoretical Sun that moves uniformly about the Earth and the actual Sun , with nearly uniform motion.

$$y(t) = l(t) - mt$$

From this definition of $y(t)$ the solution for the longitude is available.

$$l(t) = y(t) + mt$$

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[3] The Canon truncates values. Adding the values together yields the truncated value of the half year.

[4] Symmetry may be used to get the summer to fall and fall to winter outcomes.

where

- $l(t)$ is the actual longitude of the Sun at time t.
- m is the mean speed of the Sun as comupted by early Chinese astronomers., In degress per day, $m = 0.98565°/\text{day}$, In du per day, $m = 1.0$ du/day. The explanation of the du follows.
- mt is the position of a theoretical Sun in uniform motion.

One detail can't be omitted. Chinese astronomers did not measure their angles in degrees. They used a unit known as the du. One du = 0.98563 degrees, which is approximately Guo Shoujing's value for the daily mean movement.[5]. Think of 1 du as approximately the mean angle that the Sun progresses every day, which is slightly less than a degree.The unit of y in the cubic equation is in du, not degrees. Similarly the value of m should be presented in du per day. This has the advantage that one day represents one du.

Using the shadow measurements as well as night time measurements, Guo Shoujing constructed biweekly values of l and y. He then interpolated the values of y to arrive at his cubic equation. Qi Luqian provides the hint and information from *confirms* this. Below is Sivin's translation of the relevant passage from Qi Luqian.

We used the corrected ch'i of the Four Standard Points, establishing rising and descending limits on this basis. We set up interpolation to yield motion parts for each day, beginning and end, maximal differences, and accumulated degrees. These were more accurate than in ancient times.

The ch'i (qi in modern pinying) refers to two week time frames. The Four Standard points are the two solstices and the two equinoxes. From the first sentence one infers that using their set of observations, Guo Shoujing's team of astronomers established the biweekly longitudes of the Sun. This compilation follows traditional Chinese standards and his use of the word "corrected" most likely refers to corrections of previous works.

From the longitude, one could easily establish the difference value, y. This yields biweekly values of y. With regards to the time frame we consider, the first entry is precisely at the time of the winter solstice and entries continue every two weeks until they encompass the spring equinox.

The second sentence states that the team of astronomers developed an interpolation that yields the daily change in longitude, the difference between the movement of one du and the actual movement, and the actual longitude. Note that from an interpolation of the biweekly values of y, one could establish the daily values (or values at any time).

As noted above Sivin's translation of the *Canon* fully comports with Qi Luqian's passage and our interpretation. Tables with entries giving the outputs that Qi Luqian refers to are present in the *Canon*.

Nowhere does Qi Luqian explain how the astronomers executed the interpolation. This has stirred a lot of conjecture that seems to have settled on a methodology that relies upon more recent inventions, difference equations. Difference equations are front runners to calculus. No coherent explanation that transforms the difference equations into the cubic equations that are present in the *Canon* accompany the claim that difference equations were central to Guo Shoujing's method. It is a feasible approach, but there are many lengthy steps involved that require a considerably higher computational burden than the alternative that we propose. Furthermore, even if the complex computations were perfectly executed, the method introduces many opportunities for round-off errors that accumulate. We do not believe that Guo Shoujing could have obtained his established accuracy of forecast using finite difference methods.

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[5] The origin of the du goes back to an early Chinese valuation of the length of a year at 365.25 days. Astronomers set the average daily movement in longitude as the du. The annual movement in longitude is $360°$. To get the average daily movement divide 360 by 365.25. The result is one unit of du in degrees.

Perhaps it was only the intraday values that Qi Luquan refers to where Guo Shoujing employed a difference equation. The same objections for discarding this approach apply to intraday data as well.

Our approach is our own response to the following questions. What is the simplest method to determine a cubic equation that fits a series of data points? Was this method known to Chinese astronomers?

To address the first question, suppose we have a cubic polynomial. Suppose we further wish to assure that at time $t = 0$, the value of y is also zero. This follows from the convention that sets the longitude of the Sun to zero when the Sun is at the winter solstice. In this case, the cubic polynomial has the form

$$y(t) = at + bt^2 + ct^3$$

Using a table of three actual observations of y taken at three different times, we can set up three different equations and solve for the values of the constants.

Was this method known to Chinese astronomers? Of course. This was standard practice that Chinese applied to astronomy as well as other problems. Chinese mathematicians were ahead of their western counterparts in developing and solving equations of this form.

To apply this method, there is a choice to be made. Guo Shoujing had the biweekly data points to use as his observations. There are seven such points. From these seven, Guo Shoujing must select three. Every such choice yields different values for the constants.

The table below gives eight days marking the beginning of a two week period along with the value of y. Day 0 marks the winter solstice.

Day	**y**
0	0
14	0.66958
28	1.23762
42	1.69903
56	2.04869
70	2.28151
84	2.39237
98	2.37344

Table 4.2: Historical Measurements of the Tropical Year

To determine the constants, a, b, c three observations must be chosen. It would be natural to spread the selections throughout the period rather than clustering them together. The choice of days 28, 56, and 84, would be natural. (Day 0 is already used to set the constant term in the cubic equation to 0.) Below are the equations that result from the selection. We leave it to the most enthusiastic readers to solve these equations.[6]

$$\begin{aligned} 28a + 28^2 b + 28^3 c &= 1.23762 \\ 56a + 56^2 b + 56^3 c &= 2.04869 \\ 84a + 84^2 b + 84^3 c &= 2.37344 \end{aligned}$$

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[6] Using symmetry, the Fall to Winter observations may also be considered.

We conjecture that Guo Shoujing used a trial and error approach. His team made several choices and compared the results as predicted by the cubic with actual observations at different data points. In the end, the choice of three points that best fit the data was selected, or possibly an averaging technique was applied. The result is the previously mentioned cubic equation, $a = 0.051332$, $b = -0.000246$, and $c = -0.00000031$.

4.4 Guo Shoujing, the Data Scientist

Long before the term "data science" was coined, Guo Shoujing applied many of its core principles in his astronomical work. This section examines how his methodical approach to modeling solar motion can be understood through the modern lens of data science, as outlined in Chapter 2. By identifying a well-defined problem, constructing a parametric model, collecting observational data, and refining his equations based on empirical evidence, Guo Shoujing demonstrated a scientific rigor that mirrors today's best practices in data analysis.

Define the problem.

The broader problem of creating a calendar has many subproblems. This section constraints itself to the problem of determining the difference value, y(t).

Propose an input-output parametric model of the system.

Using symmetry, Guo Shoujing reduces the problem to modeling half of the Sun's orbit. Recognizing behavioral differences on each side of the spring equinox, Guo Shoujing develops two separate equations. One applies to the time frame between the winter solstice and spring equinox. The other applies to the time frame between the spring equinox and summer solstice. The equation for the period starting from the winter solstice has the following form.

$$y(t) = at + bt^2 + ct^3$$

The input is t, the elapsed time since the Sun passed through the winter solstice. The output is $y(t)$, the difference between the actual longitude and an orbit with uniform motion of one du per day.

The parameters that need to be fit to data are the coefficients, a, b, and c.

Guo Shoujing constructs a similar cubic for the time frame passing from the spring equinox to the summer solstice. By applying symmetry, he extends the two equations to cover the full annual orbit.

Identify the required data.

A set of observations indicating the passage of time from the winter solstice, along with the corresponding values of y, is necessary.

Collect and organize data as inputs and outputs.

For the training set, Guo Shoujing needs a minimum of three observations in order to solve for the three coefficients. Having more than three observations provides Guo Shoujing with the opportunity to develop different cubic polynomials by making different choices and test each cubic's performance across the entire training set. It appears that Guo Shoujing selected seven observations, each spaced precisely two weeks apart.

The project collected data across many sights over a period of around 15 years. Observations over these 15 years likely provided additional data from which to validate the model.

Define a metric that quantifies the error between model predictions and observed outputs.

The available sources contain no references to the methods used to assess the accuracy of the cubic equation; once again we are reduced to conjecture. A likely metric is the summed absolute errors. For each observation one determines the absolute value of the difference between the observation and the cubic equation yielding the absolute error. Then one sums over all the absolute errors. This would be a measurement of how well the cubic equation performs.

Apply an optimization routine to adjust the parameters and minimize the error.

Optimization routines in the modern sense were unknown to Guo Shoujing. A method that was available is compare and contrast. Each set of three observations provides its own unique cubic equation. This would have allowed Guo Shoujing to compare the summed absolute error among different cubic polynomials and choose the best one.

Validate results against additional data.

As noted above, the project had an abundance of data from which to validate the cubic equation selected in the preceding step. Qi Luqian's writings indicate that the team of astronomers routinely cross-checked their models against extensive observational data. It is most likely that the cubic equation underwent this scrutiny.

4.5 Final Thoughts

Guo Shoujing's approach exemplifies the hallmarks of data science long before the term existed: careful problem definition, model construction, data collection, and iterative refinement. Though limited by the tools and mathematical techniques of his time, his ability to construct parametric models and validate them against empirical observations demonstrates a rigorous and systematic mindset. Within the framework outlined in Chapter 2, Guo Shoujing emerges not just as an astronomer, but as an early practitioner of data-driven modeling—a data scientist in both spirit and method.

Guo Shoujing wrestled with the need to arrive at the best parameters that fit the data. He did not have the tools that were available to Gauss, who arrived at a most elegant solution as described in the chapter *Flattened: Conquering the Data*. Before moving on to Gauss' method, the next chapter, *Kepler's Wars: Mars on Earth* presents Kepler's role in tipping humanities perspective from a geocentric universe, with its ethnocentric overtones, to a heliocentric model with far broader possibilities. Kepler faces the same problem as Guo Shoujing, fitting a parametric model to a multitude of observations.

On a final note, there are similarities between the method of cubic splines and that of designing and balancing an assembly line. Both processes require splitting up something (a curve or set of tasks) into sections, optimizing across each section, and fusing the results together. Chapter 8, *Upended: Henry Ford and the Industrial Transition* presents the balancing of an assembly line.

4.6 Summary Poem: Timing the Heavens

In Xingtai born 'neath northern skies,
Where dynasties fell and empires rise,

A boy named Guo, with gifted mind,
To stars and streams his thoughts aligned.

By Li Zhi's side he came to learn,
Of ancient texts and heavens' turn—
Where numbers ruled and stars were charted,
And water's path through stone was parted.

Mongols came with sweeping change,
The Song collapsed within their range.
Yet Guo stood firm, his knowledge keen,
To serve the state through works unseen.

He built the canals, made waters flow,
And helped an empire's order grow.
But greatest still, his gaze turned high—
To stars that marched across the sky.

He placed his gnomons tall and true,
Observed the noonday shadow's cue.
He gathered data, year by year,
To mark the solstice drawing near.

And here begins the tale we know—
Where science meets the data flow.
For what he did, in ancient guise,
Would modern minds soon recognize.

He framed a question—clear and tight:
"When will the Sun reach peak in flight?"
He chose a model, cubic, clean,
To trace the Sun's path, smooth between.

Then gathered data—wide and deep,
From observatories where monks would keep
The time, the arc, the solar glow—
Their scrolls became his input row.

He trained his model, three points tight,
Then tested others through the night.
With trial and error, scores he ran,
To minimize error as best he can.

Validation? Yes, he knew—
That truth must hold for data new.
Across fifteen years and distant lands,
His model stood, precise and grand.

Today we write in Python script,
With cloud-based logs and dashboards flipped.
But still we follow steps he knew—
Define, collect, and model through.

We tune parameters, just like he,

With loss functions and MSE.
We cross-validate, we test and train—
Guo did the same, without the name.

So laugh, if “Teaching Timing” seems
A phrase unfit for modern dreams.
But to the fields, it rang so true—
A cosmic clock for all to view.

Now Guo’s bright name in stone may lie,
But Shoushi lives beneath the sky.
And through its data-driven grace,
We see the past in our own place.