Chapter 3

Aristarchus of Samos: The Redeemed Revolutionary

3.1 A Revolution on Hold: The Heliocentric Universe

Aristarchus of Samos (c. 310 BCE – c. 230 BCE) was a Greek astronomer and mathematician best known for proposing a heliocentric model of the solar system—placing the Sun, rather than the Earth, at the center of the known universe. Although this revolutionary idea would not gain serious traction until over 1,800 years later with Copernicus, Aristarchus laid crucial foundational work in understanding celestial mechanics and the relative sizes and distances of celestial bodies.

Aristarchus made significant strides in both astronomy and mathematics. Among his most notable accomplishments were:

- The earliest known proposal of a heliocentric model, asserting that the Earth orbits the Sun and rotates on its axis.
- A geometric method for estimating the relative sizes and distances of the Sun and Moon, presented in his work *On the Sizes and Distances of the Sun and Moon*.
- Contributions to understanding the lunar phases and solar eclipses through the use of trigonometry and geometry.

His heliocentric hypothesis was far ahead of its time, standing in stark contrast to the prevailing geocentric model advocated by Plato and later formalized by Ptolemy.

Aristarchus was born shortly after the death of Alexander the Great in 323 BCE. Alexander's empire fragmented into rival Hellenistic kingdoms, each kingdom ruled by one of Alexander's generals. Among these, the Ptolemaic dynasty in Egypt distinguished itself through its ambitious patronage of arts, philosophy, and science. Under Ptolemy I Soter and his successors—most notably Ptolemy II Philadelphus—Alexandria emerged as a cultural and intellectual epicenter. Central to this transformation was the establishment of the Library and Museum of Alexandria, twin institutions that became the intellectual heartbeat of the Greek world.

The Library of Alexandria was more than a vast collection of scrolls—it was the inheritor and curator of the Greek philosophical tradition. The educational model it adopted stemmed from classical precedents, particularly the Lyceum of Aristotle and the Academy of Plato. The Library housed texts on mathematics, astronomy, medicine, physics, and metaphysics, reflecting a commitment to systematic learning and dialectical inquiry.

The scholarly processes included the preservation, translation, comparison, and commentary of texts. This systematized approach helped harmonize diverse philosophies such as Platonism, with its emphasis on mathematical abstraction and ideal forms; Aristotelianism, rooted in empirical classification and natural motion; and Hellenistic philosophies like Stoicism and Epicureanism, which examined the cosmos in rational, naturalistic terms.

Integral to the vitality of Alexandrian scholarship was royal patronage. Scholars residing at the Museum were often granted stipends, housing, and the freedom to pursue knowledge without material concerns. This model encouraged intellectual audacity, providing a rare space where theorists could test radical ideas—even those that ran counter to prevailing orthodoxy. Yet such freedom had limits. Patronage came with expectations, and intellectual speculation was often tolerated only so long as it did not undermine ideological or theological consensus.

It is within this context that Aristarchus of Samos likely found his intellectual footing. Born on the island of Samos—a place steeped in scientific and mathematical heritage thanks to Pythagoras and his school—Aristarchus would have benefited from an early exposure to the Ionian tradition of rational inquiry. Although Samos had no formal academy during Aristarchus' time, its cultural environment and proximity to centers like Miletus and Ephesus suggest he had access to significant philosophical and mathematical influences.

The surviving work of Aristarchus, *On the Sizes and Distances of the Sun and Moon*, reveals familiarity with geometric methods associated with Euclid and others likely preserved in Alexandrian collections. His trigonometric reasoning, reliance on observational geometry, and engagement with celestial scales reflect the depth and rigor typical of the Alexandrian milieu. Though no direct textual reference connects Aristarchus to the Library or Museum, the sophistication of his methods and the alignment of his intellectual concerns with Alexandrian priorities strongly suggest that he was either a member of, or deeply influenced by, the scholarly environment there.

Several avenues make Aristarchus' inclusion in Alexandria's scholarly community plausible. He may have been invited by Ptolemaic authorities after gaining recognition for his early work. Alternatively, he might have journeyed to Alexandria independently, as many ambitious thinkers of his time did, and earned patronage through the demonstration of exceptional intellectual ability. The Ptolemies occasionally granted support based on merit, and Aristarchus' contributions—especially his heliocentric hypothesis, however radical—may have been recognized as valuable, if controversial, within that community.

In sum, while we lack definitive biographical records placing Aristarchus in Alexandria, the circumstantial evidence—the nature of his work, the philosophical context it responds to, and the structural opportunities available at the Library—support the conjecture that he was, at least for a time, part of the great intellectual enterprise unfolding there.

Aristarchus' heliocentric theory, which placed the Sun at the center of the known universe and the Earth in motion around it, was radically divergent from the dominant geocentric worldview of his time. While mathematically sound and philosophically provocative, the theory failed to gain traction among Greek scholars, who were firmly entrenched in the belief that Earth must occupy the central and most important position in the cosmos. One might say that cosmic egocentrism is as much a human flaw as a scientific obstacle — a theme that would rear its head again in the trials of Galileo nearly two millennia later.

A notable and rather vehement reaction came from the Stoic philosopher Cleanthes (c. 330 – c. 230 BCE), who is said to have called for Aristarchus to be indicted for impiety. According to Plutarch in his dialogue *On the Face in the Moon*, Cleanthes argued that Aristarchus' proposal to move the Earth from its central, motionless place violated both the philosophical norms and religious sentiments of the time. To suggest that the Earth spun on its axis and orbited the Sun was not just a scientific hypothesis — it was, to some, a heretical offense against the perceived cosmic order.

Despite the hostility (or perhaps because of it), Aristarchus' work lived on in references by later thinkers, most notably Archimedes, who discussed his heliocentric model in *The Sand Reckoner*. The mistake of Archimedes' life is that he rejected Aristarchus. His reasoning was that the stars' positions in the sky remained constant. If the Earth was circling the sun, the positions would vary. Archimedes was unable to imagine the vastness of the universe that results in imperceptible shifts in the stars' positions. And so, Aristarchus' model would be largely ignored until the Renaissance. The astronomer Claudius Ptolemy, who lived in the 2nd century CE, would later formalize the geocentric model into a complex but widely accepted mathematical system in his *Almagest*, featuring deferents, epicycles, and equants[1] Ptolemy's Earth-centered cosmos became doctrine for over a thousand years, a legacy so resilient it required nothing short of a scientific revolution to overturn — a revolution Aristarchus had already imagined.

In modern times, Aristarchus is recognized as a scientific visionary. Though his contemporaries failed to embrace his cosmic demotion of Earth, his willingness to challenge orthodoxy set a precedent for future generations. His use of geometry and observation in celestial theory marked a pivotal moment in the history of science — a moment met not with acclaim, but with suspicion and scorn. History, it seems, has always been slow to yield to those who dare to place the Sun, not humanity, at the center of things.

3.2 Aristarchus' Heliocentric Model and His Treatise on Celestial Sizes and Distances

The only surviving work of Aristarchus of Samos, titled *On the Sizes and Distances of the Sun and Moon*, provides a tantalizing glimpse into the logic that may have led him to propose a heliocentric model of the cosmos. Although the treatise itself does not explicitly state that the Earth revolves around the Sun, its conclusions strongly imply such a view. Aristarchus observed that the Sun must be far larger than the Earth and Moon, and from this, it seemed natural to place the largest body at the center of motion. This inference, supported by his geometric reasoning and comparative analysis, may have led him to adopt a heliocentric perspective, which he reportedly articulated in a now-lost work mentioned by Archimedes.

Aristarchus' treatise reflects the axiomatic deductive approach that the Greeks introduced and dominates technical writing throughout history. Indeed the style is familiar to any individual that reads the current mathematical or physics literature. The treatise conclusively proves that the Sun is greater in diameter than the Earth and provides a range for the ratio of the two diameters.

Aristarchus' methods demonstrate his mastery of geometry and attention to subtlety. He fills in all details leaving nothing to dispute. The details and subtle points, whether materially influential or not, make for a challenging read of an elegant analysis. We are more modest. For the purpose of presenting the elegance of Archimedes' arguments, we remove some of the subtlety and detail.

Aristarchus' proof follows three main steps:

1. Determine the relative distances to the Sun and Moon, expressed as a ratio.
2. Determine the relative sizes of the Sun and Moon, expressed as a ratio.
3. Determine the relative sizes of the Sun and Earth, expressed as a ratio.

[1] The deferent is a reference circle around the sun. A point on the deferent revolves about the sun and the Earth revolves about the point through a circle of relatively small radius. Revolutions of the earth about the point are the epicycles. The equant regulates the point's speed along the deferent. The equant is another point between the deferent and the sun about which the angular vspeed of the deferent point is constant. This causes the speed of the deferent point to vary.

We follow these steps as well.

Ratio of the distances to the Sun and Moon

Nature presents opportunities to yield her secrets, but one must be a keen observer, recognize the opportunity, and seize the moment. The presence of a half Moon permits the astute observer to determine the ratio of the distance of the Sun to that of the Moon. The three bodies form a triangle During a half moon, the angle between the Sun and Moon and Earth is 90 degrees. A measurement of the Moon-Earth-Sun angle allows one to configure and analyze a right triangle that is similar to nature's the Sun, Earth, Moon, Earth triangle. With this concept, from his "office", Aristarchus can determine the relative distances to the Sun and Moon without ever leaving his office.[2]

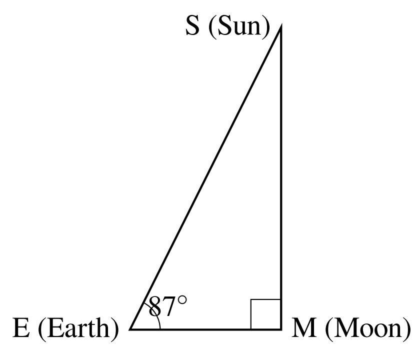

Figure 3.1: Right triangle formed by the Sun, Earth, Moon (not to scale)

Using the triangle $\triangle EMS$, with right angle at the Moon (M), and angle $\angle SEM = 87°$, Aristarchus reasoned that:

$$\frac{\text{Distance to Sun}}{\text{Distance to Moon}} = \frac{ES}{EM} = \frac{1}{\cos(87°)} \approx 19.1$$

The calculation above relies upon access to a calculator or computer; in this case, ChatGPT provides the required value of $\cos(87°)$. Not only did Aristarchus not have access to a computer, trigonometry had not yet been formalized and there were no trigonometric tables for his convenience. In his treatise, Aristarchus performs his own calculations to estimate the value $\cos(87°)$. These are difficult calculations that demonstrate Aristarchus' skill.

The Greeks were aware that irrational numbers exist. When confronted with a number which may have been irrational, a common practice (not a rule) was to bound the value between two rational numbers. Aristarchus adopts this common practice and determines:

$$\frac{1}{18} < \cos(87°) < \frac{1}{20}$$

from which one concludes:

$$18 < \frac{SE}{ME} < 20$$

[2]By referencing Aristarchus' working space as his office, we are stealing a quip from Voltaire who mocked Maupertuis and his expedition to Lapland. The quip is once again mentioned in Chapter 5.

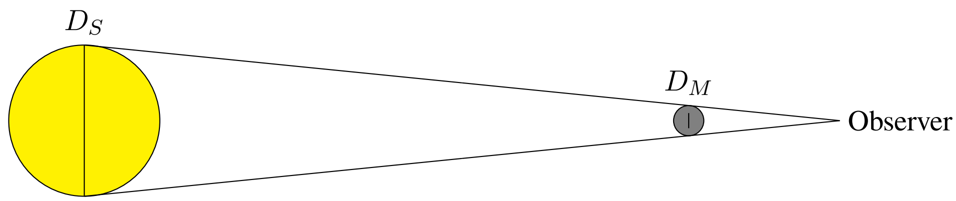

Figure 3.2: Solar eclipse geometry. The Moon lies between the Earth and Sun and appears the same size as the Sun when viewed from Earth. D_S is the diameter of the Sun and D_M is the diameter of the Moon.

Ratio of the Sizes of the Sun and Moon

Nature presents the gift of a solar eclipse; the perceptive individual uses this gift to determine the relative sizes of the Sun and Moon. During a total solar eclipse, the Moon briefly covers the face of the Sun, plunging day into night and creating a moment of awe and revelation. This single observation allows one to compare the diameters of the Sun and Moon, provided their relative distances are known.

In this moment of cosmic drama, the Moon plays the part of the jealous inferior. Gazing upon the glory of the Sun, the Moon, driven by envy, dares to veil it — though in doing so, it too vanishes from view. This bold act lasts only an instant. The brevity of the event reveals a critical truth: the Sun and Moon appear to be of equal size in the sky when viewed from Earth.

This equality of apparent sizes is evident in the Figure 3.2. In the figure, there are two similar equilateral triangles, each with their apex at the Earth bound observer. The bases of the triangles are the respective diameters of the Sun and Moon. Similarity of the triangles allows one to conclude that the ratio of the actual diameters of the Sun and Moon must equal the ratio of their distances from the observer. That is,

$$\frac{\text{Diameter of Sun}}{\text{Diameter of Moon}} = \frac{\text{Distance to Sun}}{\text{Distance to Moon}}$$

Using the earlier result from Aristarchus' analysis of the half moon,

$$18 < \frac{ES \text{ (Distance Earth to Sun)}}{EM \text{ (Distance Earth to Moon)}} < 20$$

he concluded that

$$18 < \frac{D_S \text{ (Diameter of Sun)}}{D_M \text{(Diameter of Moon)}} < 20$$

This ratio is critical in Aristarchus' argument, as it allows him to compare the Sun and Earth in the final step of his reasoning.

Ratio of the Sizes of the Sun and Earth

Just as the solar eclipse unveils the proportional kinship between the Sun and Moon, so too does nature offer a counterpart — the lunar eclipse. This celestial alignment, where the Earth steps between the Sun and Moon, serves as a mirror to the solar event and provides a new opportunity for discovery. If the solar eclipse is a gift, dramatic and fleeting, then the lunar eclipse is its serene reflection, longer in duration and equally revealing.

In a lunar eclipse, the Earth's shadow is cast upon the face of the Moon. The attentive observer notes that the Earth's shadow is not a sharp beam, but a soft-edged cone, whose circular cross-section dwarfs the Moon it envelopes. Through this spectacle, one may discern a critical relation: the ratio of the diameters of the Earth and the Sun. The shadow's geometry, informed by the size of the Earth and the scale of the Sun's rays, encodes the final step in Aristarchus' method.

Here, again, nature has staged an elegant alignment — a grand symmetry to the solar eclipse — that allows the perceptive mind to extract a profound geometric truth from a fleeting play of light and shadow. The result completes the triad of celestial proportions: Sun to Moon, Sun to Earth, and soon, Earth to Moon: D_S/D_M, D_E/D_M, and D_S/D_E.

The ratio $D_S/D_M = ES/EM = \cos(87°)$ is already known. Aristarchus uncovered the remaining proportions, D_E/D_M, and D_S/D_E. using the geometric shapes formed during a lunar eclipse as displayed in Figure 3.3.

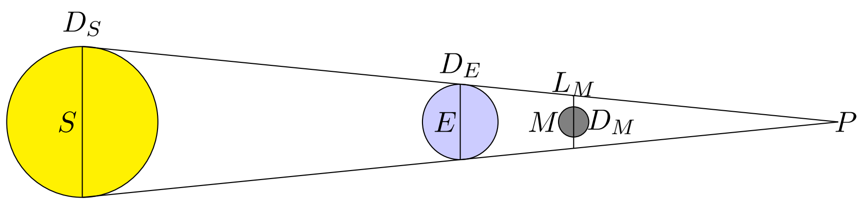

Figure 3.3: Three similar triangles formed during a lunar eclipse. Here, D_S is the diameter of the Sun, D_E is the diameter of the Earth, D_M is the diameter of the Moon, and L_M is the leg of the smallest triangle passing through the Moon.

The diagram above illustrates three key isosceles triangles that form during a lunar eclipse. The bases of the triangles are the diameters of the Sun (D_S) and Earth (D_E) as well as a line segment (L_M) that passes through the diameter of the Moon (D_M). Connecting the endpoints of each of the bases to the point P where the Earth's shadow vanishes yields three similar triangles.

By convention we set:

$$L_M = \alpha D_M$$

where the constant α is subsequently determined.

Bisecting the bases of the isosceles triangles in Figure 3.3 gives three additional similar triangles (Figure 3.4). Using these triangles and his analytic skills, Archimedes uncovers a truth that his peers found uncomfortable, $D_S/D_E >> 1$.

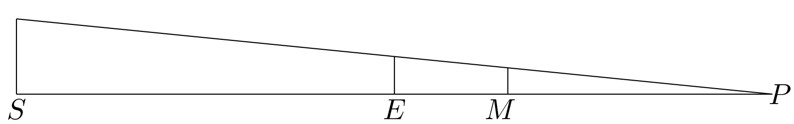

Figure 3.4: Three similar right triangles created by bysecting the bases of the triangles in Figure 3.3.

We start by determining D_E/D_M. This is not only interesting in itself, but is also used to determine D_S/D_E.

$$
\begin{aligned}
\frac{D_E}{D_M} =& \alpha \frac{D_E}{L_M} = \alpha \frac{EP}{MP} = \alpha \frac{SP - SE}{MP} \\
&= \alpha \left(\frac{SP}{MP} - \frac{SE}{MP} \right) \\
&= \alpha \left(\frac{D_S}{L_M} - \frac{SE}{MP} \right) && \text{similar triangles} \\
&= \frac{D_S}{D_M} - \alpha \frac{SE}{MP} \\
&= \frac{1}{\cos(87^\circ)} - \alpha \frac{SE}{MP} \\
&= \frac{1}{\cos(87^\circ)} - \alpha \frac{ES}{EM} \frac{EM}{MP} \\
&= \frac{1}{\cos(87^\circ)} - \alpha \frac{1}{\cos(87^\circ)} \frac{EM}{MP} \\
&= \frac{1}{\cos(87^\circ)} \left(1 - \alpha \frac{EM}{MP} \right) \\
&= \frac{1}{\cos(87^\circ)} \left(1 - \alpha \frac{EP - MP}{MP} \right) \\
&= \frac{1}{\cos(87^\circ)} \left(1 - \alpha \left(\frac{EP}{MP} - \frac{MP}{MP} \right) \right) \\
&= \frac{1}{\cos(87^\circ)} \left(1 - \alpha \left(\frac{D_E}{L_M} - \frac{MP}{MP} \right) \right) && \text{similar triangles} \\
&= \frac{1}{\cos(87^\circ)} \left(1 - \left(\frac{D_E}{D_M} - \alpha \right) \right)
\end{aligned}
$$

Simplifying the final result above gives the ratio D_E/D_M.

$$
\begin{aligned}
\frac{D_E}{D_M} &= \frac{1}{\cos(87^\circ)} \left(1 - \left(\frac{D_E}{D_M} - \alpha \right) \right) \\
\left(1 + \frac{1}{\cos(87^\circ)} \right) \frac{D_E}{D_M} &= \frac{1}{\cos(87^\circ)} (1 + \alpha) \\
\frac{D_E}{D_M} &= \frac{\frac{1}{\cos(87^\circ)} (1 + \alpha)}{1 + \frac{1}{\cos(87^\circ)}}
\end{aligned}
$$

which gives the result:

$$
\frac{D_E}{D_M} = \frac{1 + \alpha}{1 + \cos(87^\circ)}
$$

Note that:

$$
\frac{D_M}{D_E} = \frac{1 + \cos(87^\circ)}{1 + \alpha}
$$

We are approaching the finish line.

$$\begin{aligned}\frac{D_S}{D_E} &= \frac{D_S}{D_M}\frac{D_M}{D_E} \\ &= \frac{1}{\cos(87^\circ)}\frac{1+\cos(87^\circ)}{1+\alpha} \\ &= \frac{1+\frac{1}{\cos(87^\circ)}}{1+\alpha}\end{aligned}$$

To cross the finish line, we need the value of α. Let's accept Aristarchus' value, $\alpha = 2$. Aristarchus plucked this value from a personal observation noting that just as the Moon completely entered the Earth's shadow, it was half way along its journey through the shadow. Therefore the length L_M is twice that of the moon's diameter, D_M. The next section discusses this and other measurements. But with this value of α we can find Aristarchus' result.

Recall Aristarchus' calculation:

$$18 < \frac{1}{\cos(87^\circ)} < 20$$

Inserting these bounds along with $\alpha = 2$ into the expression D_S/D_E provides on the ratio D_S/D_E.

$$\frac{19}{3} < \frac{D_S}{D_E} < 7$$

This final calculation leaves no doubt. The Sun is significantly larger than the Earth.

3.3 The Measurements

In any scientific investigation—whether you're modeling the climate, predicting stock markets, or figuring out how far away the Sun is—there's one rule that never changes: the quality of your conclusions depends entirely on the quality of your data. It's the first commandment of data science, often summed up with a bit of snark: "garbage in, garbage out." Even the most brilliant model can't salvage faulty input. That was as true in ancient Greece as it is today.

This section turns our attention to the observational data—the raw measurements Aristarchus performed. His analysis, outlined in the previous section, was built not only on abstract geometry and logic, but also on concrete observations of the sky. And while the tools available to him were little more than sticks, shadows, and careful eyes, the data they yielded formed the backbone of what was, for its time, a radical and astonishingly insightful vision of the cosmos.

To fully establish the scale and structure of the solar system as Aristarchus conceived it, three key observational inputs were required:

1. The angle between the Moon and the Sun at the moment of half-moon, which Aristarchus estimated to be approximately 87°. This measurement constrains the relative distances from Earth to the Moon and to the Sun.

2. The apparent equality in the angular sizes of the Sun and Moon as seen from Earth. This supports the assumption that they subtend roughly the same angle in the sky, allowing a direct comparison of their actual sizes once their distances are known.

3. The parameter α, which relates the diameter of the Moon to the length of its path through Earth's shadow during a lunar eclipse. This relationship offered Aristarchus a means of estimating the Moon's size—and, by extension, the sizes of the Earth and Sun.

When compared to modern values, Aristarchus' estimates show significant deviations—particularly the $87°$ angle, which we now know to be closer to $89.85°$. These discrepancies were likely the result of the observational challenges of the time: limited instruments, no optics, imprecise angular tools, and the ever-present interference of the atmosphere. Still, it's worth considering whether some values were influenced not just by the constraints of measurement, but by the social, rhetorical, or philosophical aims of the treatise itself. We explore that possibility in what follows.

If we substitute Aristarchus' original measurements with modern values, his geometric reasoning yields results remarkably close to reality. This makes Aristarchus more than an ancient astronomer—he was, in effect, one of the earliest data scientists. He demonstrated how even imperfect observations, if guided by sound reasoning, can reveal deep and lasting truths about the universe. The pages ahead explore how these observations might have been made, and what they reveal about both the cosmos—and the mind that first tried to measure it.

Measuring the Angle at Half-Moon

For Aristarchus, the most critical observation is the angle between the Sun, Earth, and the Moon at the moment of half-moon (or first and third quarter). This angle, measured from Earth, directly determines the relative distances to the Sun and Moon. Figure 3.1 depicts the angle. The relative distances are extremely sensitive to the measurement when the angle is close to $90°$. Aristarchus estimated this angle at $87°$—just three degrees short of a right angle. In modern measurements, the true value is closer to $89.85°$. Aristarchus' measurement error of less than $2°$ produces a huge error in the relative distances. Using Aristarchus. measurement: $D_S/D_M = 19.107$. Using the modern measurement: The measurement error yields an error that is off by a factor of 20.

Consider what the measurement actually involves. At half-moon, the line from the Earth to the Moon is at a right angle to the line from the Moon to the Sun. To determine the angle between the Earth–Moon and Earth–Sun lines, one would need to measure the apparent separation in the sky between the Sun and the Moon at that moment.

A common tool of the day that Aristarchus may have employed for such a task was the *dioptra*, an early Greek sighting instrument—a sort of proto-sextant. The dioptra consists of a flat base supporting a pivoting semicircular (or sometimes circular) arc, over which a rotating sighting arm—called the *alidade*—is mounted. The alidade pivots around a fixed point at the center of the arc. Thin vertical sighting vanes at both ends of the alidade—typically narrow plates or rods with a pinhole or vertical slit—allow for accurate alignment with a target.

To measure the half-Moon angle, the observer would first carefully align the entire plane of the semicircle with the celestial plane defined by the Sun and Moon. This step required the base of the dioptra to be tilted and rotated so that both celestial bodies lay in the same observational plane as the instrument.

Once aligned, the alidade was rotated to sight the Moon. The observer would look through the pair of vanes on the alidade and adjust it until the Moon appeared precisely through both. Because the line of sight through the alidade passed through both vanes and intersected the center of the arc (either via a fixed central vane or by design), the observer could record the corresponding angular position.

Without moving the base or reorienting the semicircle, the observer would then rotate the alidade to sight the Sun, again aligning it visually through the sighting vanes. The new position was marked the same way. The angular separation between these two marks on the semicircle was the angle between the Earth–Moon and

Earth–Sun lines.

Whether the arc was pre-graduated in degrees or marked relatively is uncertain. The Greeks had access to angular subdivisions inherited from Babylonian sexagesimal systems, but it is likely that Aristarchus used proportional or geometric methods rather than absolute degree readings.

Figure 3.5: A reconstructed diagram of the dioptra, showing the central fixed sighting vane, the movable alidade with twin sight vanes, and the graduated semicircular arc.

The most difficult, if not impossible operation in the above procedure is the alignment of the alidade with the Sun. Accurate alignment would necessitate the observer sighting the sun directly through the vanes of the alidade. Do not expect any volunteers for that task. An alternative would make the use of two sighting poles. One pole creates a shadow while the other marks the tip of the shadow. After setting the two poles, the observer would remove the shadow-casting pole and place the dioptra at its location.The observer assures that the fixed point of the dioptra is at the previous location of the top of shadow producing pole. Using the shadow tip as a stand-in for the Sun in the procedure above gives an angle that, when adjusted by 180 degrees, yields the desired half-moon measurement.

We do not know the exact procedure that results in the angle that Aristarchus reports. We do know that similar measurements were very accurate. For example, Eudoxus' presentation of the geocentric universe, which predates Aristarchus, includes an accurate description of the tilt of Earth's polar axis. In a geocentric description, the tilt is the angle that the polar axis makes with the plane about which the Sun revolves about the Earth. An accurate description of the tilt requires an accurate measurement of the angle between the North Star, Earth, and Sun [3]. This measurement is very similar to the measurement required by Aristarchus. The accuracy of Eudoxus' astronomical descriptions suggests that the technical skills necessary for Aristarchus angular measurements were indeed available to him. Yet even by the standards of his time, the error in the half-Moon angle Aristarchus reports is surprisingly large.

Following the descriptions of the remaining two observations, we present some conjectures as to why Aristarchus did not use a more accurate measurement.

Measuring the Angular Sizes of the Sun and Moon

To determine how far away the Sun and Moon are, Aristarchus also needed to know how large they appear in the sky—their *angular size*. This refers to the angle each body appears to span when observed from Earth, not their actual diameters. A full Moon or the visible disk of the Sun might seem vast to the eye, but in angular terms, both subtend a surprisingly small angle: about half a degree.

In principle, measuring this angle is a straightforward geometric problem. Imagine placing a rod with a small hole in it at a known distance from your eye. You move the rod forward or backward until the disk of the Sun or Moon just fills the hole. Knowing the distance to the rod and the size of the hole, you can calculate the angular size using basic trigonometry. This is similar to how a modern transit or sextant works.

In practice, however, one encounters the same difficulty faced when attempting to align the dioptra's alidade with the Sun—no one can look directly at the Sun without risking permanent eye damage. Fortunately, nature occasionally provides a workaround: the solar eclipse. During a total solar eclipse, the Moon passes directly in front of the Sun and—at least from certain locations—completely obscures it.

[3] This angle is not directly measurable, but must be measured component-wise. A full description is beyond the scope of this book.

Let us imagine two possible scenarios. First, if the angular size of the Sun were notably greater than the moon; the moon would never dominate the sun and there would not be even a moment in time of complete darkness. Alternatively, if the Sun's angular size was notably smaller than that of the moon, then the Earth would be encompassed in darkness for a noticeable duration of time. To these two scenarios, let's add a third. What if the Sun and Moon's angluar sizes were just about equal? Then for those areas of the Earth plunged into darkness, the duration of total darkness would be but an instant in time.

It is the third scenario that observers of total eclipses report. Whether Aristarchus himself witnessed such an event is unknown. However, total eclipses left a powerful impression on those who did. For many, the sudden darkness evoked fear and superstition. Reports of the phenomenon were widely known in antiquity. Herodotus, for instance, recounts that the philosopher Thales of Miletus predicted a solar eclipse that brought a war between the Lydians and the Medes to an abrupt halt. Such moments, though frightening to the general populace, offered scientifically minded observers like Aristarchus valuable insight. The fleeting instant of totality suggested a close match in the angular sizes of the Sun and Moon.

Modern measurements confirm this assessment. The Sun and Moon subtend angular sizes of approximately $0.53°$ and $0.52°$ respectively[4]. The Sun's size is slightly larger on average, but the difference—about $0.01°$—is small. Unlike the significant error in Aristarchus's half-moon angle, his use of equal angular sizes for the Sun and Moon is a reasonable and remarkably good approximation for his time.

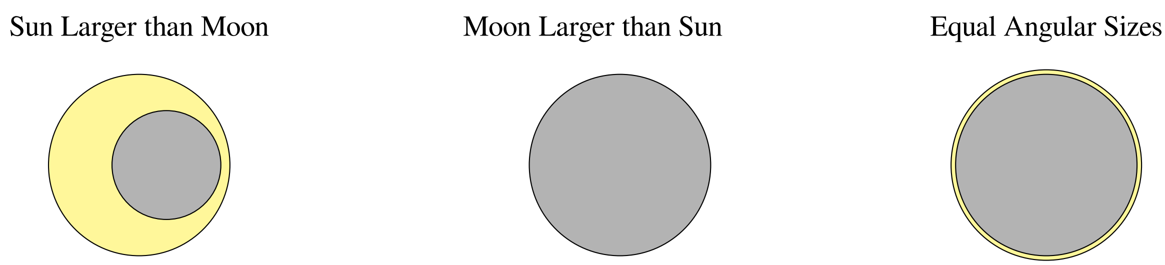

Figure 3.6: Three eclipse scenarios as viewed by Earth Observer. Left: Larger Sun, no period of darkness; center: Smaller Sun, extended period of darkness; right: Similar Size, Darkness for a moment.

Measuring the Length of the Moon's Pathway Through the Earth's Shadow

During a lunar eclipse, the Moon passes through the Earth's shadow, creating a striking and observable event. Ancient astronomers, including Aristarchus, saw in this moment an opportunity to study the geometric relationships among the Sun, Earth, and Moon. One particularly significant aspect is the length of the Moon's pathway through the shadow. This pathway, in relation to the Moon's diameter, provides a key parameter, denoted as α, that can be used to estimate the relative sizes of the celestial bodies.

The parameter α is defined as the ratio between the total length of the Moon's path through the Earth's shadow and diameter of the Moon.

$$\alpha = \frac{\text{Moon's diameter}}{\text{Length of Moon's path through the Earth's shadow}}$$

There are two plausible methods for determining this ratio. The first uses a dioptra to measure the two angles that yield α: the angle that the Moon subtends and the angle that the Moon's path through the Earth's shadow

[4]The angular sizes of both the Sun and Moon vary slightly due to the elliptical shapes of their orbits. The Sun's angular size ranges from about 31.6 to 32.7 arcminutes, while the Moon's ranges from about 29.3 to 34.1 arcminutes.

subtends. These two angles are proportional to the lengths defining α, and so it follows that their ratio equals α.

An alternative to direct measurement of the angles using a dioptra is to consider two corresponding time measurements. The first measurement is the time difference between the Moon's complete entry into the Earth's shadow and its complete exit from the Earth's shadow. This time measurement is proportional to the length of the Moon's path through the Earth's shadow. The second measurement begins when the Moon initiates its exit from the Earth's shadow and ends at the instant the Moon completely emerges from the Earth's shadow.This second time difference is proportional to the diameter of the Moon. The parameter α equals the ratio of the first time difference to the second time difference.

The alternative approach requires reliable methods of tracking time. Among the most significant instruments available to the Greeks was the water clock, or *clepsydra* (literally "water thief"). This device consisted of a vessel—typically bronze or ceramic—with a small hole near the bottom, allowing water to drain at a steady rate. Time could be measured either by the fall of the water level in an outflow clepsydra or the rise in an inflow version, which collected water from another source.

By the 3rd century BCE, engineers like Ctesibius of Alexandria had developed sophisticated versions of the clepsydra, complete with float regulators to ensure a more constant flow, and sometimes even mechanical indicators or sound alerts. These enhancements made the water clock one of the most advanced timekeeping devices of the ancient world.

In civic life, particularly in the Athenian law courts, clepsydrae were used to limit the length of speeches. Each speaker in a legal proceeding was allotted a fixed amount of time, measured by the volume of water in the clock. Once the water ran out, the speech had to end. According to some sources, litigants occasionally attempted to manipulate the timing—either by stopping the flow temporarily during interruptions or by relying on the imprecision of older clepsydrae. An anecdote recounted by the orator Demosthenes hints at this tension: he once complained that his opponent had wasted valuable time with irrelevant remarks, thus letting the clepsydra drain unfairly.[5]

Aside from its use in civic life, the astronomers around the time of Aristarchus successfully employed the clepsydrae toward scientific endeavors. For example, the Babylonian-influenced Greek astronomer Hipparchus, active in the 2nd century BCE, used a clepsydra to measure the duration of lunar eclipses with the aim of refining the length of the synodic month (the time between full Moons). During a lunar eclipse, he timed the interval from the Moon's first contact with the Earth's shadow to its final emergence using a clepsydra. By comparing this duration across multiple eclipses and correlating it with observed lunar phases, Hipparchus was able to estimate the average duration of the synodic month with remarkable precision—within a few seconds of the modern value.

Just as the instruments and techniques of the day were up to the task of providing an accurate measurement of the half-Moon angle, they were also adequate to provide an accurate measurement of α. For unknown reasons, the value that Aristarchus uses does not meet the standards of his era. Modern measurements place α at 2.6. Aristarchus sets α to 2, which differs from the modern value by over 27 percent.

About those errors

Aristarchus' work is a masterpiece. He supports an unorthodox and outright iconoclastic truth with an irrefutable argument. The attention to detail and analytic skill is equally irrefutable. In terms of detail, one point thus far not mentioned is that Aristarchus' analysis includes a term that accounts for the curved path of the

[5] ChatGPT's image generator created Figure 3.7. There are obvious the flaws that the epilogue discusses.

Figure 3.7: The clepsydrae in court.

Moon through the Earth's shadow. This term's influence on his final calculation of the ratio of sizes between the Sun and Earth is immaterial. It doesn't even affect the result by 0.01 percent. By contrast his use of $87°$ as the half-Moon angle affects the ratio by over 2000 percent. What does this say about Aristarchus and how can we account for those measurement errors? Below, we give some conjectures.

On the eve of World War I, Russian officers detained a team of German experimentalists who had within their possession an assortment of telescopes and cameras. The German team's destination was a village in Russia that was predicted to fall within the blackout zone of Solar eclipse. The suspicious Russian officers had a hard time believing the team's explanation that they were there to observe whether or not starlight initiating behind the Sun would bend about the Sun's gravity as described by Einstein's theory. Instead they went with their gut feeling that the equipment supported a spying operation and imprisoned the experimentalists.

The modern physics community splits into several groups. Among them are theoreticians and experimentalists. Individuals who cross between these groups are rare. While Einstein's proposed the experiment that would later confirm his theory of relativity, he never for an instant thought of performing the experiment himself. That would be left to the experimentalists.

The theoretical side of physics relies upon the axiomatic deductive method. This method is so seductive that specialists in fields where its application confronts significant challenges devote their lives and stake their claims on this method. The fields of economics[6] and law[7] come to mind[8].

The axiomatic deductive method is indeed seductive. This book's human author finds tremendous beauty in absolute truths that one arrives at using axiomatic deduction. The human author relates to anyone else who finds the same beauty in the truths. This human author believes that Aristarchus was firmly in this camp as evidenced by his phenomenal treatise. Aristarchus was a theoretician, not an experimentalist.

Let's take this idea one step further. Possibly, Aristarchus never made any of the measurements he proposes. The experimentalist is a tinkerer. The experimentalist encounters a downpour of unforeseen difficulties. It takes a combination of brilliance and experience to overcome the difficulties and successfully perform an experiment. Beyond brilliance, it takes patience, persistence and a lifestyle commitment as experiments are performed and perfected through ad nauseam repetition.

Rather than taking actual measurements that would have been accurate, Aristarchus may have imputed the measurements from other available scholarly works. Alternatively, he may have glanced at the sky with an outstretched thumb, and guessed. Afterall, these are not bad guesses from the an outstretched thumb.

What would the final results have been if Aristarchus had accurate measurements? As noted above, Aristarchus' analysis is impeccable. With accurate measurements, Aristarchus' results would have comported with today's modern values, $D_S/D_M \approx 389$ and $D_S/D_E \approx 109$.

Does this diminish Aristarchus' work. This is a matter of judgment in which we firmly judge, no. Our judgment reflects Aristarchus' discovery of the truth, the Sun is much larger than the Earth; a fact that Aristarchus' contemporaries found upsetting. The worst of his errors, the half Moon angle, is conservative. Archimedes demonstrates that even with his undervalued choice of $87°$, the Sun is significantly larger than the Earth. If his contemporaries were to confront their emotional rejection with reason, they would have to conclude that the angle is an understatement and the Sun is in fact much larger than the Earth and that Aristarchus' estimates are conservative.

Nevertheless, the degree to which the ratios significantly differ from reality demonstrates the importance of data integrity.

3.4 Aristarchus the Data Scientist

While Aristarchus of Samos lived over two millennia before the term "data science" was coined, his approach to understanding the cosmos remarkably anticipates the logic and structure of modern analytical methodology. In a previous chapter, we outlined the six fundamental steps that characterize the data science process. These steps provide a framework not only for modern machine learning and statistical analysis, but also for evaluating historical works of scientific inquiry through a contemporary lens.

Aristarchus' method for estimating the relative sizes and distances of the Sun, Earth, and Moon can be naturally mapped onto each of these six steps. In doing so, we uncover a compelling argument that Aristarchus' process, though constrained by the observational tools of his era, fits the mold of a model-driven, data-informed analysis long before such language existed.

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[6]The concept of a 'free market' is set as an axiom even though it does not reflect reality.

[7]The legal profession is somewhat obsessed with the method. The legal curriculum devotes itself to the Socratic method. Additionally, the Legal profession regards the constitution as a framework of axioms and prior court decisions as theorems. Subsequent decisions are often considered as theorems that must not be violated in future decisions, lest the system become inconsistent.

[8]The challenges of strict application of the axiomatic deductive method to fields where axioms are unclear or may not apply to reality explain why there are so many contradictions in these fields. Maybe it's a topic for another book.

Define the problem

The first step in the data science process is to clearly articulate the problem to be solved. For Aristarchus, the problem was both profound and deceptively simple: *What are the relative sizes of the Sun and Moon compared to the Earth?*

Propose an input-output parametric model of the system

The second step in the data science process is to translate the real-world problem into a mathematical or computational model. The model presents itself in three layers. Outputs from one layer act as inputs to subsequent layers. A previous section, *Aristarchus' Heliocentric Model and His Treatise on Celestial Sizes and Distances*, gives a detailed description of the model. Below, we list the inputs and outputs of each layer.

Layer 1: Relative Distances of the Celestial Bodies

Input:

- θ — the half-Moon angle, i.e., the angle between the Moon and the Sun as seen from the Earth when the Moon appears half illuminated.

Outputs:

- $\frac{ES}{EM}$ — the ratio of the distances from Earth to Sun and Earth to Moon.

This model uses elementary trigonometry to compute the distance ratio from the triangle formed by Earth, Moon, and Sun during the half-Moon phase.

Layer 2: Relative Size of the Sun to the Moon

Input:

- $\frac{ES}{EM}$ — the ratio of distances from Earth to the Sun and Moon, calculated from Model 1.

Outputs:

- $\frac{D_S}{D_M}$ — the ratio of the diameter of the Sun to that of the Moon.

This model assumes that the Sun and Moon subtend the same angular size from Earth during a solar eclipse. Since angular size is proportional to the ratio of physical size to distance, and their angular sizes are equal, the physical sizes must scale with their distances.

Layer 3: Relative Size of the Sun to the Earth

Inputs:

- $\alpha = \frac{L_M}{D_M}$ The ratio of the pathlength of the moon as it travels through the Earth's shadow during an eclipse to the diameter of the moon.
- $\frac{ES}{EM}$ — the ratio of distances from Earth to the Sun and Moon, calculated from Model 1.
- $\frac{D_S}{D_M}$ — the ratio of the diameter of the Sun to the Moon, calculated from Model 2.

Outputs:

- $\frac{D_E}{D_M}$ — the ratio of the diameter of the Earth to that of the Moon.

- $\frac{D_S}{D_E}$ — the ratio of the diameter of the Sun to that of the Earth.

This model uses geometric reasoning from the configuration of the Sun, Earth, and Moon during an eclipse.

Identify the required data

The parameters fall into two categories: fixed and fitted. Fixed parameters are those that are initially set and remain constant, often based on theoretical reasoning, observational assumptions, or approximations. Fitted parameters, on the other hand, are those that are initially unknown or estimated and are determined through calculations or optimization processes that best align the model with observational data. The fixed parameters are those that identify the required data.

Fixed Parameters:

- θ — the angle between the Sun and Moon as observed from Earth during the half-Moon phase.
- α — The ratio of the length of the moon's path as it travels through the Earth's shadow during an eclipse to the diameter of the moon.

Fitted Parameters:

- $\frac{ES}{EM}$ — the ratio of the Earth-Sun distance to the Earth-Moon distance.
- $\frac{D_S}{D_M}$ — the ratio of the diameter of the Sun to the diameter of the Moon.
- $\frac{D_S}{D_E}$ — the ratio of the diameter of the Sun to the diameter of the Earth.

Aristarchus' models present an instance in which the inputs are fixed parameters and outputs are fitted parameters. Calculations upon the inputs lead to the outputs. Within the general data science framework, this is not the way things always are.

Outputs can be fixed parameters and one may wish to calculate the inputs so that the model fits the outputs. That is to say the outputs are part of a data stream This book explores such examples in later chapters.

Collect and organize fixed data as inputs and outputs

As noted above, the fixed data are all inputs. Aristarchus uses two inputs.

Fixed Parameters, all inputs

- $\theta = 87^\circ$
- $\alpha = 2$

Define a metric that quantifies the error between model predictions and observed outputs

This step is necessary for the cases in which outputs are fixed parameters and the objective is to fit inputs to match outputs. Because Aristarchus' model directly computes outputs from the fixed inputs, this step is unnecessary.

Apply an optimization routine to adjust the parameters and minimize the error

The goal of this step is to minimize the difference between the input equations and the outputs by adjusting the fitted parameters. In this case, one simply plugs the inputs into Aristarchus' equations and computes the outputs. The result is that there is no error between the input equations and the outputs.

Step 7. Validate results against additional data. Aristarchus has no need for this step. He applies the model to a single problem and using his only available data, answered the question.

3.5 Final Thoughts

Aristarchus is not merely a practitioner of data science, but a pioneering founder. The success of his parametric model demonstrates the strength of the mathematical approach with incremental enhancements that have led to among other things, ChatGPT.

Aristarchus' inputs are the fixed parameters of his parametric model that follow directly from his measurements. The next few chapters follow an arc in which parameters are not determined by direct measurement, but chosen to fit a dataset of many observations. Evolution toward this aim culminates with Legendre and Gauss' least squares method which the chapter *Flattened: Conquering the Data* describes.

Before presenting Legendre and Gauss' conquest, the next two chapters illustrate the use of multiple observations as sources for determining the parameters and validating parameters in a parametric model.

One final point to note is the architecture of Aristarchus' model. The structure is a set of layered results in which outputs of one layer act as inputs to subsequent layers. This architecture portends that of neural networks as described in Chapter 10, *AI, Neural Networks, and the Connectors*.

The structure is not unique to Archimedes' model and neural networks. As just one example, the entire body of theory for mathematics through the axiomatic deductive process relies upon results from one theorem acting as inputs to generate new theorems. Further examples are given throughout the book.

3.6 Summary Poem: Herald of the Sun, Herald of Data Science

In days when stars obeyed the Earth,
A voice from Samos gave new birth.
Aristarchus dared to claim,
The Sun, not Earth, should hold the flame.

He prized clear form over precise gain,
Let geometry alone explain.
He sought not numbers to impress,
But rules that stars themselves confess.

The Moon at half, a triangle made,
From angle cast, a truth displayed:
The Sun stood distant, far and wide—
A beacon none could now deride.

In shadowed Moon, eclipse would show,
The Earth's round bulk in silent glow.
From curve of shade and angle spun,
He saw the larger size of Sun.

Though Cleanthes called for blasphemy,
And scoffed at such audacity,
He held his course, though scorned and mocked,
While dogma stood, his thought was blocked.

His treatise stands, though terse and bold,
In structured form the heavens told.
The Sun's great size, he set in place,
To rule the center, not just space.

But time would pass, and minds would stall,
Till Copernicus recalled it all.
Two thousand years would cloak the spark
Once lit by quiet Aristarch.

He asked the skies, then sought the light,
Observed the arcs from noon to night,
Passed through six steps with data's grace—
A process modern minds embrace.

Though ancient tools were all he knew,
His method bore a logic true:
Define, collect, and test with care—
A path that data minds now share.

Today, where numbers frame our view,
And charts give birth to insights new,
We trace his steps in code and sum—
The data scientist has come.

Now hailed as sage, his star ascends,
Where science, courage, vision blends.
He dared to move the Earth—and so,

He changed the course of what we know.